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Arithmetic on the Cheap: Neologicism and the Problem of the Logical Ontology

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Abstract: Scottish Neologicism aims to found arithmetic on full second-order logic and Hume's Principle, stating that the number of the *F*s is identical with the number of the *G*s if, and only if, there are as many *F*s as *G*s. However, Neologicism faces the *problem of the logical ontology*, according to which the underlying second-order logic involves ontological commitments. This paper addresses this issue by substituting second-order logic by Boolos's *plural logic*, augmented by the Plural Frege Quantifier $\mathcal{F}$ modelled on Antonelli's Frege Quantifier. The resulting theory (PHP) interprets second-order Peano arithmetic. Its ontological innocence is assessed: PHP offers an alternative that solves the problem of the logical ontology pervading Neologicism.

Key words: Neologicism, ontological commitment, second-order logic, plural logic, Hume's Principle, second-order Peano arithmetic

One of the most influential and debated orientations in abstractionism in the philosophy of mathematics is *(Scottish) Neologicism* by Hale and Wright. Neologicism aims to found second-order Peano arithmetic PA^2 on Frege Arithmetic (FA), which consists in full second-order logic (SOL) and Hume's Principle

(HP) $\#F = \#G \leftrightarrow F \approx G$,

stating that, for any concepts *F* and *G*, the number of the *F*s is identical with the number of the *G*s if, and only if, *F* and *G* are *equinumerous*—i.e., there is a bijection between the *F*s and the *G*s.[1] FA is consistent and interprets PA^2.

Neologicism has faced extended philosophical scrutiny since its inception. In this paper, I will address a less explored issue, which Hale and Wright label *the problem of the logical ontology*. In short, this issue concerns the ontological commitment of the underlying SOL.

In §1, I will discuss the problem of the logical ontology. In §2, I will present an axiomatic system (PHP) for Neologicism relying on Boolos's plural logic (PL) and the Plural Frege Quantifier $\mathcal{F}$ modelled on Antonelli's Frege Quantifier in Antonelli (2010), and a plural formulation of FA—the interpretation of PA^2 in PHP is provided in the Appendix. In §3, I will investigate the problem of the logical ontology with respect to PHP, and argue that PHP solves it. In §4, I will provide concluding remarks.

1. THE PROBLEM OF THE LOGICAL ONTOLOGY

Famously, Quine disparaged SOL as set theory in disguise, but, even without subscribing to Quine's view, qualms may persist concerning the ontology SOL requires—regardless of the kind of entities

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it is committed to. Even dismissing Quine's identification of the values of second-order variables with sets, one might still be uncomfortable with assuming the existence of other kinds of second-order entities, such as properties or concepts, and might demand an explanation of what they are and why we should accept them. This is the issue Hale and Wright label *the problem of the logical ontology*.[2]

Wright (2007) argues that the ontological commitment of SOL, *if any*, is not elicited by second-order quantification *per se*, but rather SOL inherits it from the existence of the semantic values of predicate letters in first-order logic.

According to Wright, there are (at least?) two stances towards SOL. One might hold that first-order predicate letters are ontologically committed, and pass that commitment on to second-order quantification. Wright calls this approach the *range-of-(existing)-entities* view, which falls prey to the problem of the logical ontology.[3]

According to the second stance, one can reject that first-order predicate letters are ontologically committed, and, therefore, second-order quantification is too. This is the view Wright (2007) labels *Nominalist Neutralism*, and is entrenched in a view of quantification as a *device of generalisation* from singular to general *thoughts*—i.e., from thoughts concerning singular entities to thoughts expressed by quantification over them.

Wright touches upon two objections to Neutralism: (i) likely, real analysis could not be recovered because of a lack of sufficient singular thoughts to generalise over; (ii) in this view it is not clear how to provide values for genuinely impredicative predicates, i.e., predicates expressed by formulæ containing second-order quantifiers, whose instances may be predicates expressed by formulæ containing second-order quantifiers.[4]

Issues (i) and (ii) are tightly connected. Consider FA: general second-order thoughts are either denumerably or uncountably many. If the former, full FA—and full PA^2—would be renounced. If the latter, most of them would be ungraspable. Finally, it remains unclear whether the opponent of second-order ontological commitment would readily accept the existence of a range of (intelligible) true thoughts. At the very least, it can be questioned whether commitment to true thoughts is any more intelligible or ontologically innocent than commitment to second-order entities.

2. REWRITING THE AXIOMS

The above may not be knockdown arguments against Wright's Neutralism, but they do highlight its problematic status, allowing to explore alternatives.

This section will provide an axiomatization of HP relying on Boolos's plural logic PL. My goal will be to defend the ontological innocence of the resulting system.

2.1. Language and Semantics

The language L of *Plural HP* (PHP) is a language of PL containing:

(i) singular individual variables x, y, z, . . . ;

(ii) plural individual variables xx, yy, zz, . . . , *plurally* varying over the first-order domain;

(iii) the logical constants $\neg$, $\rightarrow$, $=$;

(iv) existential quantifiers $\exists$ for each kind of variables;

(v) the constant $\prec$;

(vi) the abstraction function #;

(vii) the *Plural Frege Quantifier* $\mathcal{F}$.

The inspiration for $\mathcal{F}$comes from Antonelli's Frege Quantifier **F** in Antonelli (2010).[5] **F** is a primitive first-order quantifier binding singular variables and taking two formulæ φ, ψ as arguments, stating that *there are no more φs than ψs*.[6] The quantifier $\mathcal{F}$ is an adaptation of Antonelli's **F** bent to the use of plural resources: $\mathcal{F}$ is a primitive of *L* binding singular individual variables and taking *plural* formulæ as arguments.

Besides *x*, *y*, *z*, the terms of *L* are:

(viii) singular terms of the form #*xx*,

reading informally 'the number of the *xx*s.'

Besides existential formulæ of the form $\exists x\varphi$ and $\exists xx\varphi$,[7] and formulæ containing logical connectives, the well-formed formulæ of *L* are:

(a) $x = y$;

(b) $x \prec xx$;

(c) $\mathcal{F}x[(x \prec xx), (x \prec yy)]$.

Formulæ of kind (b), i.e., plural formulæ, read intuitively '*x* is among the *xx*s.' Formulæ of kind (c), in which $\mathcal{F}$ binds a singular variable *x* and takes two plural formulæ as arguments, read informally 'there are no more *xx*s than *yy*s.'

The semantic clauses for the non-$\mathcal{F}$-formulæ of *L* are: (i) Boolos's *plural objectual* clauses for plural existential quantification; (ii) a usual semantics in terms of *sequences* for first-order existential quantification. Let *D* be a non-empty first-order domain. The satisfaction clauses for the above formulæ are provided in terms of an assignment relation *R*, assigning zero or more individuals of *D* to plural variables, and a sequence σ of individuals of *D*.[8]

In Antonelli (2010), the interpretation of **F** hinges on the existence of a mapping among the extensions of formulæ, but as for $\mathcal{F}$ we can use pluralities, since the 'extension' of $x \prec xx$ just is the values of *xx*, if any, that *R* assigns to *xx* in a model $\mathcal{M}$. Thus, for any assignment of objects of *D* to the singular and plural variables, the 'extension' of $x \prec xx$ in $\mathcal{M}$ is the plurality aa^{xx}_{R}.[9] Finally, the satisfaction clause and models for $\mathcal{F}$ can be provided:

> Definition 1. For any assignments σ, *R* of objects in *D* to the singular and plural variables, $\mathcal{M} \vDash_{[\sigma, R]} \mathcal{F}x[(x \prec xx), (x \prec yy)]$ iff there is a one-one mapping *f* from the plurality aa^{xx}_{R} into the plurality bb^{yy}_{R}.

> Definition 2. A model for $\mathcal{F}$ is a structure $\mathcal{M}$ containing a non-empty first-order domain *D*, an interpretation of the non-logical constants, and the collection $\mathfrak{F}$ of all one-one functions *f*: $aa \rightarrow bb$ with $dom(f) = aa$, and $rng(f) \subseteq$ bb, for $aa, bb \subseteq D$.

In Antonelli (2010), general models for **F** are construed, satisfying several closure conditions.[10] $\mathcal{F}$ satisfies such conditions, since it is interpreted on the collection of all injections between subpluralities of *D*. The most crucial condition for present purposes can be formulated in terms of pluralities:

> (2.1) if $f \in \mathfrak{F}$, *f* is one-one from *aa* to *bb*, and $a \prec aa$ and $b \prec bb$, then there is a $g \in \mathfrak{F}$ such that *g* is one-one from $aa \cup \{a\}$ to $bb \cup \{b\}$,

stating that, if the *aa*s can be injected into the *bb*s, then the *aa*s plus an individual *a* (not among the *aa*s) can be injected into the *bb*s plus an individual *b* (not among the *bb*s).

2.2. Axioms

Plural quantification is governed by the following *Plural Comprehension axiom*:

(PL-CA) $\exists xx \forall y(y \prec xx \leftrightarrow \varphi y)$,

for φ not containing xx free.[11]

Notably, the only model-theoretical property of $\mathcal{F}$ necessary to interpret PA^2 is expressed by condition (2.1)—see the Appendix. L contains unrestricted plural resources, so condition (2.1) can be captured axiomatically, unlike in Antonelli (2010). To this end, PHP contains also a further axiom governing $\mathcal{F}$—'$x \prec xx^{-y}$' abbreviates '$x \prec xx \wedge x \neq y$':

($\mathcal{F}$-Ax) $\forall xx, yy(\forall x, y(x \prec xx \wedge y \prec yy \wedge \mathcal{F}z[(z \prec xx^{-x}), (z \prec yy^{-y})]) \leftrightarrow \mathcal{F}z[(z \prec xx), (z \prec yy)])$,

stating that x is among the xxs and y is among the yys and the xxs minus x can be injected into the yys minus y if and only if the xxs can be injected into the yys. The left-to-right direction of $\mathcal{F}$-Ax captures condition (2.1). The right-to-left direction states a rather natural condition, since $\mathcal{F}$ is injective.

Equinumerosity can be defined in the obvious way:

Definition 3. $xx \approx yy \leftrightarrow_{def} \mathcal{F}x[(x \prec xx), (x \prec yy)] \wedge \mathcal{F}x[(x \prec yy), (x \prec xx)]$.

Finally, HP is formulated plurally:

(HP_{PL}) $\forall xx, yy(\#xx = \#yy \leftrightarrow xx \approx yy)$.[12]

2.3. Frege Arithmetic

Frege Arithmetic is recovered in a plural fashion. By PL-CA, it holds:

(1) $\exists xx \forall y(y \prec xx \leftrightarrow y \neq y)$.

Call this plurality *oo*. Then, define 0 as $\#oo$.

Famously, Frege's predecessor is defined as a dyadic relation:

(2) $P(x, y) \leftrightarrow_{\text{def}} \exists F \exists u(Fu \wedge x = \#F^{-u} \wedge y = \#F)$.

Still, the *definiens* in (2) above contains no pairs at all. We can use this fact in the following *abbreviation*:

Abbreviation 1. $\textsc{pred}(x, y) \leftrightarrow_{abb} \exists xx \exists u(u \prec xx \wedge x = \#xx^{-u} \wedge y = \#xx)$.

The above must be provided as an abbreviation, rather than as a definition, since there are no pluralities of ordered pairs in PHP. For the sake of simplicity, I will call PRED 'predecessor' (and the abbreviations below 'ancestral of the predecessor' and 'weak ancestral'), but it must be kept in mind that the abbreviated formula is what is referred to.

The other notions required for recovering FA can also be abbreviated:

Abbreviation 2. $\textsc{her}(ss, \textsc{pred}) \leftrightarrow_{abb} \forall x, y(\textsc{pred}(x, y) \wedge x \prec ss \rightarrow y \prec ss)$;

Abbreviation 3. $\textsc{pred}^{\star}(x, y) \leftrightarrow_{abb} \forall ss(\forall z(\textsc{pred}(x, z) \rightarrow z \prec ss) \wedge \textsc{her}(ss, \textsc{pred}) \rightarrow y \prec ss)$;

Abbreviation 4. $\textsc{pred}^{+}(x, y) \leftrightarrow_{abb} \textsc{pred}^{\star}(x, y) \vee x = y$.

Finally, by a valid instance of PL-CA, the plurality of the natural numbers is introduced:

Definition 4. $x \prec \mathbb{N} \leftrightarrow_{def} \textsc{pred}^{+}(0, x)$.

Abbreviations 2, 3, and 4 do not concern arbitrary dyadic relations (though they concern formulæ in two singular variables), since these are not expressible in PL unless pairs of individuals are added to it. By contrast, each of the previous abbreviations can be unfolded in formulæ of L *not* containing notation for alleged ordered pairs of individuals. In general, there is no need for ordered

pairs in L, as long as abbreviations 2, 3, and 4 are restricted to PRED—which is what is needed to recover FA and show that PA^2 is interpretable in PHP.

In ordinary Neologicist approaches, pairs are needed only to provide a definition of $\approx$. But in L the latter is defined in terms of $\mathcal{F}$, which requires ordered pairs of *formulæ in the metalanguage*. Assuming $\mathcal{F}$ as a primitive does not make L polyadic. Hence, FA (and PA^2) can be interpreted in *monadic* PL augmented by HP_{PL} and $\mathcal{F}$-Ax.

3. THE PROBLEM OF THE LOGICAL ONTOLOGY REVISITED

The aim of this section is to investigate the problem of the logical ontology concerning PHP, by focussing on $\mathcal{F}$ and plural quantification.

Although my goal is to avoid higher-order ontological commitment, assuming $\mathcal{F}$ as a primitive introduces a commitment to functions. However, while reference to functions is involved in the semantics and model construction of $\mathcal{F}$ (and PHP),[13] those functions are *not* genuine objects of quantification in L, *pace* Quine. This sense of ontological commitment is labelled *narrow* in Florio and Linnebo (2021).

Florio and Linnebo (2021) argue that, to fully appreciate the commitments of a theory, we should rely on a *broad* conception by which "ontological commitment is tied to the presence of existential quantifiers of *any logical category* in a sentence's truth conditions" (Florio and Linnebo 2021: 164; italics in original). Therefore, to assess ontological commitment, we should examine *what the semantics of the language of a theory is committed to*—in contrast with the narrow conception, by which the ontological commitments of a theory are unveiled by the (existential) quantifiers of its language and the values of its variables. If *that* is the notion of ontological commitment one endorses, $\mathcal{F}$ and PHP are *not* commitment-free.

Still, the broad conception might be considered *too* broad. Take the Tarskian semantics for existential statements in first-order logic:

> $\exists xAx$ is true if, and only if, there is at least a x-variant sequence σ_j of a given sequence σ of individuals of D such that $\sigma_j(x) \in I(A)$,

whose right-hand side (existentially) quantifies over (infinite) sequences of individuals of D. On the broad conception, classical first-order logic is (also) committed to the existence of sequences, hence is not broadly ontologically innocent. However, if this is the case, one might argue that no (classical) theory is broadly ontologically innocent at all, and suggest that the notion of broad ontological commitment loses its edge.

The semantics underlying PL-CA is Boolos's plural semantics, by which there is no plural domain at all. However, Henkin models for PL can be construed, in which a plural domain has to be fixed, and consequently *which pluralities exist* has to be determinate, over and above mere first-order individuals. In this respect, "the role of plural logic as a philosophical tool appears substantially diminished," since "there is a precise and interesting sense in which plural logic may be said to be committing. Whether this commitment is ontological or ideological, it is a full-fledged form of commitment nonetheless" (Florio and Linnebo 2021: 168).

I contend that the objection relying on Henkin models is effective only under the broad conception of ontological commitment. Still, if the broad conception is resisted, as suggested, the existence of Henkin models does not necessarily reject the (narrow) ontological innocence of PL. By the latter, the ontological commitments of a theory are carried by what objects PL quantifies over. Since PL just quantifies over first-order individuals, PL can still be regarded as ontologically innocent, even in non-standard (plural) interpretations.

PHP solves the problem of the logical ontology by avoiding it altogether. Under the narrow reading of ontological commitment, on the one hand PL is not committed to the existence of plu-

ralities as entities of some kind in an appropriate domain, on the other the Plural Frege Quantifier is not committed to the existence of functions as objects of quantification in the language of PHP.

4. CLOSING REMARKS

I have proposed a solution to the *problem of the logical ontology*, concerning the ontological commitment of SOL underlying the Neologicist axiomatisation of FA. To secure ontological innocence, SOL is replaced by Boolos's PL.

Famously, though, PL interprets at most monadic SOL. This limitation would impact negatively the definitions necessary to recover FA, particularly the definition of equinumerosity. To address this limitation, the Plural Frege Quantifier $\mathcal{F}$ modelled on Antonelli (2010) was introduced, by which equinumerosity is defined. Notably, this augmentation does not make the language of PL polyadic. The resulting axiomatic system PHP interprets (a plural formulation of) FA, and (a plural formulation of) second-order Peano arithmetic PA^2—see below.

Last, the ontological innocence of PHP is investigated. In this regard, PHP offers an alternative solving the problem of the logical ontology pervading Neologicism.

APPENDIX: PA^2

Let us introduce singular and plural parameters $a, b, c, \ldots$ and $aa, bb, cc, \ldots$ in the language of PHP.

Theorem 1. PA1: $0 \prec \mathbb{N}$.
Proof. It follows from Def. 4.

Theorem 2. PA2: $\forall x, y, z(\textsc{pred}(x, z) \wedge \textsc{pred}(y, z) \rightarrow x = y)$.
Proof. Assume $\textsc{pred}\ (a, c) \wedge \textsc{pred}\ (b, c)$. By abbreviation 1, $\exists xx \exists u(u \prec xx \wedge a = \#xx^{-u} \wedge c = \#xx)$ and $\exists yy \exists w(w \prec yy \wedge b = \#yy^{-w} \wedge c = \#yy)$, which imply $\#aa = \#bb$ (for xx being aa, and yy being bb). By HP_{PL} and the right-to-left direction of $\mathcal{F}$-Ax, $\#aa^{-d} = \#bb^{-e}$ (for $u = d$, $w = e$) implying $a = b$.

Theorem 3. PA3: $\forall x, y, z\ (\textsc{pred}(x, y) \wedge \textsc{pred}(x, z) \rightarrow y = z)$.
Proof. Assume $\textsc{pred}(a, b) \wedge \textsc{pred}(a, c)$. By abbreviation 1, $\exists xx \exists u(u \prec xx \wedge a = \#xx^{-u} \wedge b = \#xx)$ and $\exists yy \exists w(w \prec yy \wedge a = \#yy^{-w} \wedge c = \#yy)$, which imply $\#aa^{-d} = \#bb^{-e}$ (for xx being aa, yy being bb, and $u = d$, $w = e$). By HP_{PL} and the left-to-right direction of $\mathcal{F}$-Ax, $b = c$.

Theorem 4. PA4: $\neg \exists\ x(x \prec \mathbb{N} \wedge \textsc{pred}(x, 0))$.
Proof. Assume $\exists x(x \prec \mathbb{N} \wedge \textsc{pred}(x, 0))$. By abbreviation 1 and definition of 0, $\exists x \exists u(x \prec \mathbb{N} \wedge u \prec oo \wedge x = \#oo^{-u} \wedge 0 = \#oo)$. By definition, oo is $x \neq x$, so $u \not\prec oo$, hence $\neg \exists x(x \prec \mathbb{N} \wedge \textsc{pred}(x, 0))$.

Theorem 5. PA5: $\forall xx\ (0 \prec xx \wedge \textsc{her}(xx, \textsc{pred}) \rightarrow \forall y(y \prec \mathbb{N} \rightarrow y \prec xx))$
Proof. Assume: (1) $0 \prec aa \wedge \textsc{her}\ (aa, \textsc{pred})$; (2) $b \prec \mathbb{N}$. We prove $b \prec aa$. By Def. 4 and (2), $\textsc{pred}^{+}\ (0, b)$, i.e., $\textsc{pred}^{\star}(0, b) \vee b = 0$ by abbreviation 4. Assume (3) $b = 0$: by $0 \prec aa$ from (1), $b \prec aa$ by substitution of identicals. Assume (4) $\textsc{pred}^{\star}(0, b)$, i.e., $\forall xx(\forall z(\textsc{pred}(0, z) \rightarrow z \prec xx) \wedge \textsc{her}(xx, \textsc{pred}) \rightarrow b \prec xx)$ by abbreviation 3. Assume (5) $\textsc{pred}(0, b)$. By (1), $\textsc{her}(aa, \textsc{pred})$, i.e., $\forall x, y(\textsc{pred}(x, y) \wedge x \prec aa \rightarrow y \prec aa)$ by abbreviation 2, which implies $\textsc{pred}(0, b) \wedge 0 \prec aa \rightarrow b \prec aa$. By assumption (5) and $0 \prec aa$ from (1), $\textsc{pred}(0, b) \wedge 0 \prec aa$, therefore $b \prec aa$. By the latter and (5), $\textsc{pred}(0, b) \rightarrow b \prec aa$, hence $\forall z(\textsc{pred}(0, z) \rightarrow z \prec aa)$. By the latter, $\textsc{her}(aa, \textsc{pred})$ from (1), and $\forall z(\textsc{pred}(0, z) \rightarrow z \prec aa) \wedge \textsc{her}(aa, \textsc{pred}) \rightarrow b \prec aa$ from (4), $b \prec aa$ follows.

Theorem 6. PA6: $\forall x(x \prec \mathbb{N} \rightarrow \exists y(y \prec \mathbb{N} \wedge \textsc{pred}(x, y))$.
Proof. For readability, I will use the following convention: whenever a complex formula φx corresponds to a plurality, I will construe the corresponding number-term as '$\#[x.\varphi x]$.'

The proof proceeds by induction, by a (plural) analogue of the *Lemma on Successors* in Zalta 2024: $\forall x(\textsc{pred}(x, \#[z.\textsc{pred}^+(z, x)]))$, according to which every x precedes the number of individuals weakly preceding x.

Since the operator # applies to *pluralities*, a worry concerns whether $\textsc{pred}^+(z, x)$ is one. It isn't, since in the language of PHP plural formulæ are only monadic. Still, what we need for the inductive proof is that, when substituting x by singular terms, i.e., 0 and a (or the like), $\textsc{pred}^+(z, 0)$ and $\textsc{pred}^+(z, a)$ define pluralities. And they do, since the following are valid (and well-formed) instances of PL-CA:

(3) $\exists xx\forall z(z \prec xx \leftrightarrow \textsc{pred}^+(z, 0))$,

(4) $\exists xx\forall z(z \prec xx \leftrightarrow \textsc{pred}^+(z, a))$.[14]

Inductive Base:

We have to show that $\textsc{pred}(0, \#[z.\textsc{pred}^+(z, 0)])$, i.e., $\exists xx\exists u(u \prec xx \wedge 0 = \#xx^{-u} \wedge \#[z.\ \textsc{pred}^+(z, 0)] = \#xx)$ by abbreviation 1. Let xx be $\textsc{pred}^+(z, 0)$, and $u = 0$: we have to show that $\textsc{pred}^+(0, 0)$, which holds trivially; $\#[z.\ \textsc{pred}^+(z, 0)] = \#[z.\ \textsc{pred}^+(z, 0)]$, which holds by identity; $0 = \#[z.\ \textsc{pred}^+(z, 0) \wedge z \neq 0]$. To prove the latter, we show $\neg\exists x(\textsc{pred}^+(x, 0) \wedge x \neq 0)$. By Thm. 4 and $\textsc{pred}^\star(a, b) \rightarrow \exists z\ \textsc{pred}(z, b)$,[15] $\textsc{pred}^\star(a, 0)$ does not hold. Suppose there's an x such that $\textsc{pred}^+(x, 0) \wedge x \neq 0$. By the latter and abbreviation 4, $\textsc{pred}^\star(a, 0) \vee a = 0 \wedge a \neq 0$, so $\textsc{pred}^\star(a, 0)$ must hold, *contra* the above.

Inductive Step:

We prove that $\textsc{pred}(a, \#[z.\ \textsc{pred}^+(z, a)])$, i.e., by abbreviation 1, there are some xxs and a u such that: $u \prec xx$; $a = \#xx^{-u}$; $\#[z.\ \textsc{pred}^+(z, a)] = \#xx$. For xx being $\textsc{pred}^+(z, a)$ and $u = a$, we have to show: $\textsc{pred}^+(a, a)$, which holds trivially; $\#[z.\textsc{pred}^+(z, a)] = \#[z.\textsc{pred}^+(z, a)]$, which holds by identity; $a = \#[z.\textsc{pred}^+(z, a) \wedge z \neq a]$. Assume (1) $\textsc{pred}(b, a)$, and (2) $\textsc{pred}(b, \#[z.\textsc{pred}^+(z, b)])$. By (1), (2) and Thm. 3, we get $a = \#[z.\textsc{pred}^+(z, b)]$, which implies we have to show $\#[z.\ \textsc{pred}^+(z, b)] = \#[z.\textsc{pred}^+(z, a) \wedge z \neq a]$.

To prove the latter, it suffices to prove $\forall z(\textsc{pred}^+(z, b) \leftrightarrow (\textsc{pred}^+(z, a) \wedge z \neq a))$. As shown in Zalta 2024, the left-to-right direction requires $\textsc{pred}^+(c, b) \wedge \textsc{pred}(b, a) \rightarrow \textsc{pred}^\star(c, a)$, and $\forall x(x \prec \mathbb{N} \rightarrow \neg\ \textsc{pred}^\star(x, x))$. The right-to-left requires that if $\textsc{pred}^\star(a, b)$, $\textsc{pred}(c, b)$, and Thm. 2 hold, then $\textsc{pred}^+(a, c)$.

Finally, we prove that the successor of a natural number is a natural number. Assume $a \prec \mathbb{N} \wedge \textsc{pred}(a, b)$. By Def. 4, $\textsc{pred}^+(0, a)$. By the fact that $\textsc{pred}^+(0, a) \wedge \textsc{pred}(a, b) \rightarrow \textsc{pred}^\star(0, b)$, it follows $\textsc{pred}^\star(0, b)$, which implies $b \prec \mathbb{N}$.[16]

Remark. Above, only monadic $\mathcal{F}$ was used. However, in order to define addition and multiplication, *dyadicity* is necessary. If $\mathcal{F}$ is allowed to bind two first-order variables, addition and multiplication can be defined in the language of PHP, along the lines of Antonelli 2010—see also note 6 above. Also, since additivity and commutativity of addition and multiplication follow by induction—hints to that end are in Antonelli 2010: §7, and PHP proves Thm. 5, commutativity and additivity should follow in PHP too.

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NOTES

1. Axiomatically, full SOL amounts to the unrestricted comprehension principle, $\exists F \forall x(Fx \leftrightarrow \varphi x)$, stating the existence of a concept F corresponding to any formula φ (without F free) of the language of SOL.
2. Hale and Wright 2001: 430ff.
3. According to Wright, *Hale's abundant conception of properties* faces this issue, since in this view the semantic values of second-order variables are the properties denoted by all possible predicates. From the point of view of the ontological innocence of SOL, as Wright (2007) points out, this is not a viable option.
4. See Wright 2007: §7. Wright suggests to solve (ii) by semantic grounding and generic quantification, but leaves this suggestion as an open problem. In Wright 2021, the notion of *generic* vs. instance-based quantification is deemed promising. This view, though, is not yet developed enough to apply to higher-order quantification.
5. I sincerely thank Jack Woods for bringing Antonelli's results to my attention.
6. Precisely, **F** is a *polyadic* binary quantifier binding vectors of first-order variables and taking two formulæ as arguments.
7. Universal quantifiers can be defined in the obvious way.
8. See Boolos 1998b: 80.
9. I.e., with respect to an assignment R of zero or more individuals aa to the plural variable xx.
10. See Antonelli 2010: 168.
11. Some systems of PL—see, e.g., Florio and Linnebo 2021—reject the existence of the empty plurality. In Boolos 1998c and 1998b and PHP, there is no such ban.
12. Let the first-order domain $\mathfrak{O}$ be $\omega \cup \{\aleph_0\}$; let $\wp(\mathfrak{O})$ be the domain plural variables vary over; let $\mathfrak{F}$ be the class of all injections between members of $\wp(\mathfrak{O})$; and let $\mathfrak{f}$ be a surjective function from members of $\wp(\mathfrak{O})$ to members of $\mathfrak{O}$, satisfying $\mathrm{HP}_{\mathrm{PL}}$. Though this construction is just a sketch, it is clearly a model of PHP.
13. See Def. 1 and note 12.
14. Also the formula $\textsc{pred}(x, \#[z.\textsc{pred}^+(z, x)])$ defines a plurality: $\exists xx \forall x(x \prec xx \leftrightarrow \textsc{pred}(x, \#[z.\textsc{pred}^+(z, x)]))$, so it can be plugged in Thm. 5 in the proof below.
15. Assume $\textsc{pred}^\star(a, b)$. By abbreviation 3, $\forall xx(\forall z(\textsc{pred}(a, z) \rightarrow z \prec xx) \wedge \textsc{her}(xx, \textsc{pred}) \rightarrow b \prec xx)$. By PL-CA, $\exists xx \forall x(x \prec xx \leftrightarrow \exists y\, \textsc{pred}(y, x))$. Call this plurality qq, and let it instantiate the abbreviation above: $\forall z(\textsc{pred}(a, z) \rightarrow z \prec qq) \wedge \textsc{her}(qq, \textsc{pred}) \rightarrow b \prec qq$. It's a theorem that, if a precedes c, there's a y preceding c (i.e., $c \prec qq$—hence $\forall z(\textsc{pred}(a, z) \rightarrow z \prec qq)$), from which it follows that, if a precedes c and $a \prec qq$, then $c \prec qq$. By universal introduction on a, c, the latter is $\textsc{her}(qq, \textsc{pred})$ by abbreviation 2. Hence, for all z, if a precedes z, there's a y preceding z (i.e., $z \prec qq$), and $\textsc{her}(qq, \textsc{pred})$, which yield $b \prec qq$ (i.e., $\exists z\, \textsc{pred}(z, b)$) by $\forall z(\textsc{pred}(a, z) \rightarrow z \prec qq) \wedge \textsc{her}(qq, \textsc{pred}) \rightarrow b \prec qq$ above.
16. All theorems used in Zalta 2024 to prove (1) $\textsc{pred}^+(c, b) \wedge \textsc{pred}(b, a) \rightarrow (b, a)\ \textsc{pred}^\star(c, a)$, (2) $\forall x(x \prec \mathbb{N} \rightarrow \neg\, \textsc{pred}^\star(x, x))$, (3) if $\textsc{pred}^\star(a, b)$, $\textsc{pred}(c, b)$, and Thm 2 hold, then $\textsc{pred}^+(a, c)$, and (4) the theorem $\textsc{pred}^+(0, a) \wedge \textsc{pred}(a, b) \rightarrow \textsc{pred}^\star(0, b)$, follow in PHP without PL-CA—so, they do not require that a given formula corresponds to a plurality. For the sake of conciseness, I will not prove (1)–(4), but the reader can follow Zalta 2024 in order to check that they are indeed provable in PHP mutatis mutandis—if restricted to $\textsc{pred}$, $\textsc{pred}^\star$, and $\textsc{pred}^+$.

REFERENCES

Antonelli, A. 2010. "Numerical Abstraction via the Frege Quantifier," *Notre Dame Journal of Formal Logic* 51: 161–179. https://doi.org/10.1215/00294527-2010-010

Boolos, G. 1998a. *Logic, Logic, and Logic*, ed. J. Burgess and R. Jeffrey. Cambridge, MA: Harvard University Press.

Boolos, G. 1998b. "Nominalist Platonism," in *Logic, Logic, and Logic*, ed. J. Burgess and R. Jeffrey, 73–87. Cambridge, MA: Harvard University Press.

Boolos, G. 1998c. "To Be Is to Be the Value of a Variable (or the Values of Some Variables)," in *Logic, Logic, and Logic*, ed. J. Burgess and R. Jeffrey, 54–72. Cambridge, MA: Harvard University Press.

Florio, S., and Ø. Linnebo. 2021. *The Many and the One: A Philosophical Study of Plural Logic*. Oxford: Oxford University Press. https://doi.org/10.1093/oso/9780198791522.001.0001

Hale, B., and C. Wright. 2001. *The Reason's Proper Study: Essays Towards a Neo-Fregean Philosophy of Mathematics*. Oxford: Oxford University Press. https://doi.org/10.1093/0198236395.001.0001

Wright, C. 2007. "On Quantifying into Predicate Position: Steps Towards a New(tralist) Perspective," in *Mathematical Knowledge*, ed. M. Leng, A. Paseau, and M. Potter, 150–74. Oxford: Oxford University Press. https://doi.org/10.1093/oso/9780199228249.003.0009

Wright, C. 2021. "Taking Stock: Hale, Heck, and Wright on Neo-Logicism and Higher-Order Logic," *Philosophia Mathematica* 29: 392–416. https://doi.org/10.1093/philmat/nkab017

Zalta, E. 2024. "Frege's Theorem and Foundations for Arithmetic," in *The Stanford Encyclopedia of Philosophy* (Spring edition), ed. Edward N. Zalta and Uri Nodelman. https://plato.stanford.edu/archives/spr2024/entries/frege-theorem/.